Quick Revision Guide

Class 12 Mathematics

Prepared by: Manish Khambete
PGT Maths, PM SHRI KV BNP Dewas

Reviewed by: Mrs. Gourima Dikshit
Principal, KV GAIL Jhabua

Inspiration: Shri R. Senthil Kumar
Deputy Commissioner, KVS Bhopal Region

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Chapter 1

Relations and Functions

Introduction

This chapter explores relations and functions, fundamental concepts in mathematics. We will cover various types of relations (empty, universal, equivalence), properties of relations (reflexive, symmetric, transitive), different types of functions (one-one, onto, bijective), composition of functions, and invertible functions.

Types of Relations

A relation in a set A is a subset of $A \times A$.

- **Empty Relation:** $R = \emptyset \subset A \times A$. No element of A is related to any element of A.
- **Universal Relation:** $R = A \times A$. Every element of A is related to every element of A.

Properties of Relations

Let R be a relation on set A.

- **Reflexive:** $(a, a) \in R$ for all $a \in A$.
- **Symmetric:** If $(a_1, a_2) \in R$, then $(a_2, a_1) \in R$.
- **Transitive:** If $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$, then $(a_1, a_3) \in R$.
- **Equivalence Relation:** A relation is an equivalence relation if it is reflexive, symmetric, and transitive.

Equivalence Class

Given an equivalence relation R on a set A, the **equivalence class** of an element $a \in A$ is the set of elements related to a under R, denoted by $[a]$:

$$[a] = \{x \in A \mid (a, x) \in R\}.$$

The set of all equivalence classes forms a partition of A.

Types of Functions

Let $f : X \to Y$ be a function.

- **One-one (Injective):** $f(x_1) = f(x_2) \implies x_1 = x_2$ for all $x_1, x_2 \in X$.
- **Onto (Surjective):** For every $y \in Y$, there exists at least one $x \in X$ such that $f(x) = y$.
- **Bijective:** A function that is both one-one and onto.

Counting Relations

Let A be a set with n elements.

- **Total Relations:** The total number of relations on A is 2^{n^2}.
- **Reflexive Relations:** The total number of reflexive relations on A is 2^{n^2-n}.
- **Symmetric Relations:** The total number of symmetric relations on A is $2^{\frac{n(n+1)}{2}}$.

Counting Functions ($f : A \to B$)

Let $|A| = m$ and $|B| = n$ (cardinality of sets A and B). The total number of functions depends on the relationship between m and n:

- **Total Functions:** The total number of functions from A to B is n^m
- **If $m \le n$:** The total number of injective (one-to-one) functions is $n(n-1)(n-2)\ldots(n-m+1) = \frac{n!}{(n-m)!}$.
- **If $m > n$:** There are no injective functions ($m > n$ makes injectivity impossible).
- **If $m = n$:** The total number of bijective functions (one-to-one and onto) is $n!$.

Chapter 2

Inverse Trigonometric Functions

Introduction

In Chapter 1, we learned about inverse functions. This chapter explores trigonometric functions to create invertible functions and examines their properties.

Basic Trigonometric Functions

The basic trigonometric functions are defined as follows:

- $\sin : \mathbb{R} \to [-1, 1]$
- $\cos : \mathbb{R} \to [-1, 1]$
- $\tan : \mathbb{R} - \{(2n + 1)\frac{\pi}{2} : n \in \mathbb{Z}\} \to \mathbb{R}$
- $\cot : \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \to \mathbb{R}$
- $\sec : \mathbb{R} - \{(2n + 1)\frac{\pi}{2} : n \in \mathbb{Z}\} \to \mathbb{R} - (-1, 1)$
- $\csc : \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \to \mathbb{R} - (-1, 1)$

Defining Inverse Functions

To define we need domain where it is one-to-one and onto. The range of the inverse function is then this restricted domain. The principal value branch is defined as follows:

- $\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $\cos^{-1} : [-1, 1] \to [0, \pi]$
- $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$
- $\cot^{-1} : \mathbb{R} \to (0, \pi)$
- $\sec^{-1} : \mathbb{R} - (-1, 1) \to [0, \pi] - \{\frac{\pi}{2}\}$

Adjoi The graphs of the inverse trigonometric functions can be obtained by reflecting the graphs of the corresponding trigonometric functions across the line $y = x$.

Properties

- $\sin(\sin^{-1} x) = x$, for $x \in [-1, 1]$
- $\sin^{-1}(\sin x) = x$, for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $\cos(\cos^{-1} x) = x$, for $x \in [-1, 1]$
- $\cos^{-1}(\cos x) = x$, for $x \in [0, \pi]$
- $\tan(\tan^{-1} x) = x$, for $x \in \mathbb{R}$
- $\tan^{-1}(\tan x) = x$, for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$
- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, for $x \in [-1, 1]$
- $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, for $x \in \mathbb{R}$
- $\sin^{-1}(-x) = -\sin^{-1} x$, for $x \in [-1, 1]$
- $\cos^{-1}(-x) = \pi - \cos^{-1} x$, for $x \in [-1, 1]$
- $\tan^{-1}(-x) = -\tan^{-1} x$, for $x \in \mathbb{R}$
- $2\sin^{-1} x = \sin^{-1}(2x\sqrt{1 - x^2})$, for $x \in [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$
- $2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$, for $x \in [\frac{1}{\sqrt{2}}, 1]$
- $2\tan^{-1} x = \tan^{-1}(\frac{2x}{1-x^2})$, for $|x| < 1$

Chapter 3

Matrices

Introduction

Matrices are fundamental mathematical objects represented as rectangular arrays of numbers or functions. They provide a powerful and efficient way to represent and manipulate data, especially in linear algebra and its applications. This chapter introduces the basic concepts and operations related to matrices.

Definition

Order: A matrix with m rows and n columns has order $m \times n$. We denote a matrix A as $A = [a_{ij}]_{m \times n}$, where a_{ij} represents the entry in the i-th row and j-th column.

Types of Matrices

- **Column Matrix:** A matrix with only one column ($m \times 1$). ↕
- **Row Matrix:** A matrix with only one row ($1 \times n$). ↔
- **Square Matrix:** A matrix with an equal number of rows and columns ($m = n$). ■
- **Diagonal Matrix:** A square matrix where all non-diagonal entries are zero.
- **Scalar Matrix:** A diagonal matrix with all diagonal entries equal.
- **Identity Matrix (I):** A scalar matrix with diagonal entries equal to 1. ☑
- **Zero Matrix (0):** A matrix where all entries are zero. ⊘

Matrix Operations

- **Addition/Subtraction:** Element-wise addition or subtraction (same order matrices).
- **Scalar Multiplication:** Multiply each element by a scalar k: $kA = [ka_{ij}]$.
- **Matrix Multiplication:** AB exists if the number of columns in A equals the number of rows in B. The (i, j)-th entry of AB is given by the dot product of the i-th row of A and the j-th column of B: $\sum_{k=1}^{n} a_{ik} b_{kj}$.
- **Transpose:** The transpose A^T (or A') interchanges rows and columns.
  - $(A^T)^T = A$
  - $(A + B)^T = A^T + B^T$
  - $(kA)^T = kA^T$
  - $(AB)^T = B^T A^T$
- **Inverse:** If $AA^{-1} = A^{-1}A = I$, then A^{-1} is the inverse. Only square matrices can have inverses.
  - $(A^{-1})^{-1} = A$
  - $(AB)^{-1} = B^{-1}A^{-1}$
  - $(A^T)^{-1} = (A^{-1})^T$

Special Matrices

- **Symmetric Matrix:** A square matrix A such that $A = A^T$.
- **Skew-Symmetric Matrix:** A square matrix A such that $A = -A^T$. The diagonal entries are always zero.

Chapter 4

Determinants

Introduction

Determinants are scalar values associated with square matrices. They have significant applications in solving systems of linear equations, finding areas of triangles, and determining matrix invertibility.

Order One

For a 1×1 matrix $A = [a]$, the determinant is simply $|A| = a$.

Order Two

For a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the determinant is: $|A| = a_{11}a_{22} - a_{12}a_{21}$

Order Three

For a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the determinant can be calculated by cofactor expansion along any row or column. Expanding along the first row, we have: $|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

Minors and Cofactors

- **Minor (M_{ij}):** The minor of element a_{ij} is the determinant of the submatrix obtained by deleting the i-th row and j-th column.
- **Cofactor (A_{ij}):** The cofactor of element a_{ij} is given by $A_{ij} = (-1)^{i+j} M_{ij}$.

The determinant can be expressed as the sum of products of elements of a row (or column) with their corresponding cofactors. For example, expanding along the first row: $|A| = \sum_{j=1}^{n} a_{1j}A_{1j}$.

Adjoint and Inverse

- **Adjoint (adj A):** The adjoint of matrix A is the transpose of its cofactor matrix: $(\text{adj } A)_{ij} = A_{ji}$.
- **Inverse (A^{-1}):** If $AA^{-1} = A^{-1}A = I$, then A^{-1} is the inverse of A. $A^{-1} = \frac{1}{|A|}\text{adj } A$. A matrix is invertible if and only if its determinant is non-zero.

Properties:

- $A \cdot adj(A) = adj(A) \cdot A = |A|I$
- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $|A^{-1}| = \frac{1}{|A|}$
- $|\text{adj } A| = |A|^{n-1}$ (for an $n \times n$ matrix)

Area of a Triangle

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is given by: Area $= \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right| = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$

System of Linear Equations

A system of linear equations can be expressed in matrix form as $AX = B$.

- **Unique Solution:** If $|A| \neq 0$, then $X = A^{-1}B$ gives the unique solution.
- **No Solution or Infinitely Many Solutions:** If $|A| = 0$, further analysis is needed (consider $(\text{adj } A)B$).

Additional Properties

- $|kA| = k^n|A|$ (for an $n \times n$ matrix)
- $|AB| = |A||B|$
- If A and B are symmetric matrices, then $AB - BA$ is a skew-symmetric matrix.

Chapter 5

Continuity and Differentiability

Introduction

This chapter extends the study of differentiation from Class XI, introducing crucial concepts like continuity, differentiability, and their relationship. We'll cover the differentiation of inverse trigonometric functions, exponential functions, and logarithmic functions. We'll also explore fundamental theorems in this area.

Continuity at a Point

Let f be a real-valued function defined on a subset of real numbers, and let c be a point in the domain of f. Then f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

More precisely, f is continuous at c if the left-hand limit, the right-hand limit, and the value of the function at $x = c$ exist and are equal. A function f is continuous if it is continuous at every point in its domain. For a function defined on the closed interval $[a, b]$, continuity at a means $\lim_{x \to a^+} f(x) = f(a)$, and continuity at b means $\lim_{x \to b^-} f(x) = f(b)$.

Algebra of Continuous Functions

Let f and g be two continuous functions at a real number c. Then:

1. $f + g$ is continuous at $x = c$.
2. $f - g$ is continuous at $x = c$.
3. $f \cdot g$ is continuous at $x = c$.
4. $\frac{f}{g}$ is continuous at $x = c$, provided $g(c) \neq 0$.

Every rational function is continuous in its domain (except at points where the denominator is zero). The sine and cosine functions are continuous everywhere.

Differentiability

Let f be a real-valued function and c be a point in its domain. The derivative of f at c is:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

provided this limit exists. If this limit exists, we say f is differentiable at c. If f is differentiable at every point in an interval, then f is differentiable on that interval. A function is differentiable if it's differentiable at every point in its domain. Every differentiable function is continuous, but the converse is not necessarily true. The derivative of f is denoted by $f'(x)$, $\frac{df}{dx}$, or $f'(x)$.

Chain Rule

If $y = f(g(x))$ then, $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$
This can be extended to more than two functions.

Derivatives of Inverse Trigonometric Functions

The derivatives of the inverse trigonometric functions are:

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, for $x \in (-1, 1)$
- $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$, for $x \in (-1, 1)$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, for $x \in \mathbb{R}$
- $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$, for $x \in \mathbb{R}$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$
- $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$

Exponential and Logarithmic Functions

The exponential function with base $b > 0$ and $b \neq 1$ is defined as $f(x) = b^x$. The natural exponential function is $f(x) = e^x$, where e is Euler's number. The logarithmic function with base b is the inverse of the exponential function, defined as $\log_b x = y$ if and only if $b^y = x$. The natural logarithmic function is $\ln x = \log_e x$.

Derivatives:

- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(\ln x) = \frac{1}{x}$, for $x > 0$
- $\frac{d}{dx}(a^x) = a^x \ln a$, for $a > 0$ and $a \neq 1$
- $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$, for $x > 0$ and $a > 0$ and $a \neq 1$

Logarithmic differentiation is a technique to differentiate functions of the form $y = [u(x)]^{v(x)}$ by taking logarithms of both sides.

Parametric Forms

If $x = f(t)$ and $y = g(t)$, then:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

The second derivative is given by:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$$

Differentiation Formulas

1. $\frac{d}{dx}c = 0$
2. $\frac{d}{dx}x^n = nx^{n-1}$
3. $\frac{d}{dx}f(x) + g(x) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
4. $\frac{d}{dx}f(x) \cdot g(x) = f'(x)g(x) + f(x)g'(x)$
5. $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
6. $\frac{d}{dx}\sin x = \cos x$
7. $\frac{d}{dx}\cos x = -\sin x$
8. $\frac{d}{dx}\tan x = \sec^2 x$
9. $\frac{d}{dx}\cot x = -\csc^2 x$
10. $\frac{d}{dx}\sec x = \sec x \tan x$
11. $\frac{d}{dx}\csc x = -\csc x \cot x$
12. $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
13. $\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$
14. $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$
15. $\frac{d}{dx}\ln x = \frac{1}{x}$
16. $\frac{d}{dx}e^x = e^x$
17. $\frac{d}{dx}a^x = a^x \ln a$

Chapter 6

Applications of Derivatives

Introduction

This chapter explores applications of derivatives in various fields. We will see how derivatives help us understand rates of change, find equations of tangents and normals to curves, locate maxima and minima of functions, and approximate values of quantities.

Rate of Change

If a quantity y is a function of another quantity x, i.e., $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x. If two variables x and y are functions of a third variable t, i.e., $x = f(t)$ and $y = g(t)$, then by the chain rule:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \text{if } \frac{dx}{dt} \neq 0$$

Increasing/Decreasing Functions

Let f be a function defined on an interval I. Then f is:

- **Increasing** on I if $x_1 < x_2$ in $I \implies f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$.
- **Decreasing** on I if $x_1 < x_2$ in $I \implies f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$.
- **Constant** on I if $f(x) = c$ for all $x \in I$, where c is a constant.

First Derivative Test: Let f be continuous on $[a, b]$ and differentiable on (a, b). Then:

- f is increasing if $f'(x) > 0$ for all $x \in (a, b)$.
- f is decreasing if $f'(x) < 0$ for all $x \in (a, b)$.
- f is constant if $f'(x) = 0$ for all $x \in (a, b)$.

Maxima and Minima

Let f be a function defined on an interval I, and let c be an interior point of I. Then c is:

- A point of **local maxima** if there exists an $h > 0$ such that $f(c) \geq f(x)$ for all $x \in (c - h, c + h)$ with $x \neq c$. The value $f(c)$ is the local maximum value.
- A point of **local minima** if there exists an $h > 0$ such that $f(c) \leq f(x)$ for all $x \in (c - h, c + h)$ with $x \neq c$. The value $f(c)$ is the local minimum value.
- A point of **local extrema** if it is either a local maxima or local minima.

First Derivative Test:

- If $f'(x)$ changes from positive to negative as x increases through c, then c is a point of local maxima.
- If $f'(x)$ changes from negative to positive as x increases through c, then c is a point of local minima.
- If $f'(x)$ does not change sign, then c is neither a local maxima nor a local minima (it could be a point of inflection).

Second Derivative Test: If $f'(c) = 0$ and $f''(c) < 0$, then c is a point of local maxima. If $f'(c) = 0$ and $f''(c) > 0$, then c is a point of local minima. If $f'(c) = 0$ and $f''(c) = 0$, the test is inconclusive.

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find all critical points in (a, b) (where $f'(x) = 0$ or $f'(x)$ does not exist).
2. Evaluate $f(x)$ at all critical points and at the endpoints a and b.
3. The largest value is the absolute maximum, and the smallest value is the absolute minimum.

Chapter 7

Integrals

Introduction to Integral Calculus

Integral calculus is fundamentally concerned with the inverse process of differentiation. The original motivations for integral calculus stemmed from problems of calculating areas under curves and finding functions given their derivatives. These two problems – finding antiderivatives and calculating areas – form the core of integral calculus.

Integration as the Inverse of Differentiation

If a function f is differentiable on an interval I, and its derivative f' exists at each point in I, then the question arises: given f', can we determine the function f? Functions that produce f' as their derivative are called **antiderivatives** (or **primitives**) of f. The formula representing all antiderivatives is called the **indefinite integral** of f, and the process of finding them is called **integration**. The indefinite integral of $f(x)$ is denoted by $\int f(x)\,dx$. If $F(x)$ is an antiderivative of $f(x)$, then

$$\int f(x)\,dx = F(x) + C$$

where C is called the **constant of integration** (an arbitrary constant).

Properties of Indefinite Integrals

1. $\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$
2. $\int f'(x)\,dx = f(x) + C$
3. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
4. $\int kf(x)\,dx = k\int f(x)\,dx$, where k is a constant.
5. $\int [k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x)]\,dx = k_1\int f_1(x)\,dx + k_2\int f_2(x)\,dx + \cdots + k_n\int f_n(x)\,dx$

Two indefinite integrals with the same derivative differ only by a constant.

Methods of Integration

Several techniques exist for evaluating integrals. Some common methods include:

1. **Integration by substitution:** This involves changing the variable of integration to simplify the integral.
2. **Integration using partial fractions:** This method applies to rational functions (ratio of polynomials). The rational function is expressed as a sum of simpler rational functions that are easier to integrate.
3. **Integration by parts:** This technique is useful for integrating products of functions.

Integration by Substitution

This technique transforms the integral $\int f(x)\,dx$ by substituting $x = g(t)$, so that $dx = g'(t)\,dt$, to get:

$$\int f(x)\,dx = \int f(g(t))g'(t)\,dt$$

This method is particularly useful when the derivative of a function in the integrand also appears.

Partial Fractions and Integration

Linear factor:	$\frac{1}{ax+b} = \frac{A}{ax+b}$
Repeated linear factor:	$\frac{1}{(ax+b)^2} = \frac{A}{(ax+b)} + \frac{B}{(ax+b)^2}$
Quadratic factor:	$\frac{1}{(x^2+a^2)} = \frac{Ax+B}{(x^2+a^2)}$

Integration by Parts

This method is used for integrating products of functions. If u and v are differentiable functions of x, then:

$$\int uv\,dx = u\int v\,dx - \int\left[\left(\frac{d}{dx}u\right)\cdot\int v\,dx\right]dx$$

The choice of which function to integrate first is significant. A general guideline is to choose the function whose integral is readily known as the second function. For first function, use the ILATE (inverse, log, algebaric, trig, exponential)

Indefinite Integration Formulas

1. $\int \frac{1}{x}\,dx = \ln|x| + C$
2. $\int e^x\,dx = e^x + C$
3. $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$
4. $\int \sin x\,dx = -\cos x + C$
5. $\int \cos x\,dx = \sin x + C$
6. $\int \sec^2 x\,dx = \tan x + C$
7. $\int cosec^2 x\,dx = -\cot x + C$
8. $\int \sec x \tan x\,dx = \sec x + C$
9. $\int cosec\,x \cot x\,dx = -cosec\,x + C$
10. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$
11.
12. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
13. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$
14. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
15. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log|x + \sqrt{x^2-a^2}| + C$
16. $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x + \sqrt{x^2+a^2}\right| + C$
17. $\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}(\frac{x}{a}) + C$
18. $\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log|x + \sqrt{x^2+a^2}| + C$
19. $\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log|x + \sqrt{x^2-a^2}| + C$
20. If $\int f(x)\,dx = F(x) + c$, then $\int f(ax+b) = \frac{1}{a}F(ax+b) + c$

Definite Integrals

The definite integral of a continuous function $f(x)$ over the interval $[a, b]$ is given by:

$$\int_a^b f(x)\,dx$$

The value of a definite integral is a unique number. If $F(x)$ is an antiderivative of $f(x)$, then by the second fundamental theorem of calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The numbers a and b are called the **limits of integration**, a being the lower limit and b the upper limit.

Properties of Definite Integrals

Several key properties simplify the evaluation of definite integrals:

- $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^{a+b} f(a+b-x)\,dx$
- $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$
- $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(2a-x) = f(x)$
- $\int_0^{2a} f(x)\,dx = 0$ if $f(2a-x) = -f(x)$
- $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is an even function
- $\int_{-a}^a f(x)\,dx = 0$ if $f(x)$ is an odd function

Chapter 8

Applications of Integrals

Introduction

In geometry, we learn formulas for calculating areas of various shapes. However, these formulas are inadequate for calculating areas enclosed by curves. Integral calculus provides the tools to compute such areas. This chapter explores specific applications of integrals in calculating areas under curves and between curves.

Area Under a Curve

The area A of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ (where $a < b$) is given by:

$$A = \int_a^b |f(x)|\, dx$$

If the curve is entirely above the x-axis, then $|f(x)| = f(x)$, and if it's entirely below the x-axis, then $|f(x)| = -f(x)$. If the curve is partly above and partly below the x-axis, we integrate separately for the regions above and below the axis and add the absolute values of the resulting areas.

Area Between Curves

The area A of the region bounded by two curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ (where $a < b$) is given by:

$$A = \int_a^b |f(x) - g(x)|\, dx$$

The absolute value ensures that we add the areas of regions above and below.

Areas of Regions

We can use integration to calculate the areas of regions bounded by different combinations of curves and lines. A careful examination of the given curves and lines is necessary to set up the correct integral. Common examples include regions bounded by:

- A curve, the x-axis, and two vertical lines ($x = a$ and $x = b$).
- Two curves and two vertical lines.
- A curve and a line, and two vertical lines.

The choice of integration method (vertical or horizontal strips) can significantly affect the ease of calculation.

Chapter 9

Differential Equations

Introduction to Differential Equations

A differential equation is an equation that involves an independent variable, a dependent variable, and the derivative(s) of the dependent variable with respect to the independent variable. The study of differential equations is crucial in mathematics and has broad applications in many scientific fields. This chapter focuses on first-order and first-degree differential equations.

Order and Degree

- **Order:** Order of the highest-order derivative present in the equation.
- **Degree:** If the differential equation is a polynomial equation in the derivatives, then the degree is the highest power of the highest-order derivative present in the equation. Else, the degree is not defined.

The order and degree (if defined) of a differential equation are always positive integers.

General and Particular Solutions

- **General Solution:** A solution that contains as many arbitrary constants as the order of the differential equation.
- **Particular Solution:** A solution obtained from the general solution by assigning specific values to the arbitrary constants.

Methods for Solving Differential Equations

The choice of method depends on the specific form of the equation. Common methods include:

1. **Variable Separable:** The equation can be written in the form $\frac{dy}{dx} = f(x)g(y)$. The solution is obtained by separating the variables and integrating. $\int \frac{dy}{g(y)} = \int f(x)\,dx + C$
2. **Homogeneous Differential Equations:** A homogeneous differential equation is of the form $\frac{dy}{dx} = f(\frac{y}{x})$, where f is a homogeneous function of degree 0. The substitution $y = vx$ (hence, $\frac{dy}{dx} = v + x\frac{dv}{dx}$), transforms the equation into a separable form. Similarly, if $\frac{dx}{dy} = g(\frac{x}{y})$, then substitute $x = vy$ (hence, $\frac{dx}{dy} = v + y\frac{dv}{dy}$) to convert into variable separable.
3. **Linear Differential Equations:** The equation is of the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x only. The solution involves finding an integrating factor, $I.F. = e^{\int P\,dx}$. Then, the solution is given by $y(I.F.) = \int Q(I.F.)\,dx + C$. Similarly, for $\frac{dx}{dy} + Px = Q$, where P and Q are functions of y only. $IF = e^{\int P\,dy}$, and $x \cdot IF = \int Q \cdot IF\,dy$

Chapter 10

Vector Algebra

Introduction to Vectors

Vectors are mathematical objects possessing both magnitude and direction. Unlike scalars (which only have magnitude), vectors are essential for representing physical quantities like force, velocity, and displacement. This chapter explores vector algebra, covering fundamental concepts, operations, and their applications.

Basic Vector Concepts

- **Vector:** A quantity with both magnitude and direction. Represented graphically as a directed line segment.
- **Magnitude (or length):** Denoted as $|\vec{a}|$ or $|\vec{AB}|$ or a. It's the distance between the initial and terminal points of the vector.
- **Zero Vector ($\vec{0}$):** A vector with zero magnitude. Its initial and terminal points coincide.
- **Unit Vector:** A vector with a magnitude of 1. The unit vector in the direction of $\vec{a}$ is denoted as $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.
- **Coinitial Vectors:** Vectors sharing the same initial point.
- **Collinear Vectors:** Vectors parallel to the same line.
- **Equal Vectors:** Vectors with the same magnitude and direction.
- **Negative of a Vector:** A vector with the same magnitude but opposite direction.

Direction Cosines and Ratios

Let $\vec{r} = xi + yj + zk$ be a vector.

- **Direction Angles:** The angles α, β, γ that the vector makes with the positive x, y, and z axes, respectively.
- **Direction Cosines:** The cosines of the direction angles: $l = \cos\alpha = \frac{x}{|\vec{r}|}$, $m = \cos\beta = \frac{y}{|\vec{r}|}$, $n = \cos\gamma = \frac{z}{|\vec{r}|}$. Note that $l^2 + m^2 + n^2 = 1$.
- **Direction Ratios:** Numbers proportional to the direction cosines: a, b, c such that $\frac{a}{l} = \frac{b}{m} = \frac{c}{n} = k$ for some scalar k.

Vector Addition

Triangle Law of Addition: If two vectors are represented by two sides of a triangle taken in order, then their sum (or resultant vector) is given by the third side taken in the reverse order. $\vec{AC} = \vec{AB} + \vec{BC}$.

Parallelogram Law of Addition: If two vectors are represented by two adjacent sides of a parallelogram, then their sum is given by the diagonal of the parallelogram through their common point. $\vec{AC} = \vec{AB} + \vec{AD} = \vec{AB} + \vec{BC}$.

Properties of Vector Addition:

1. Commutative: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. Associative: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
3. Additive Identity: $\vec{a} + \vec{0} = \vec{a}$

Scalar Multiplication

If $\vec{a}$ is a vector and λ is a scalar, then $\lambda\vec{a}$ is a vector collinear to $\vec{a}$. Its magnitude is $|\lambda||\vec{a}|$, and its direction is the same as $\vec{a}$ if $\lambda > 0$ and opposite to $\vec{a}$ if $\lambda < 0$. The following properties hold:

1. $k\vec{a} + m\vec{a} = (k + m)\vec{a}$
2. $k(m\vec{a}) = (km)\vec{a}$
3. $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$

Vector Components

Any vector $\vec{r}$ can be expressed in component form as $\vec{r} = xi + yj + zk$, where x, y, and z are its scalar components along the x, y, and z axes respectively, and $\vec{i}$, $\vec{j}$, and $\vec{k}$ are unit vectors along these axes. The magnitude of $\vec{r}$ is given by $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.

Section Formula

If a point R divides the line segment joining points $P(\vec{a})$ and $Q(\vec{b})$ in the ratio $m : n$, then:

1. Internally: $\vec{r} = \frac{n\vec{a}+m\vec{b}}{m+n}$
2. Externally: $\vec{r} = \frac{m\vec{b}-n\vec{a}}{m-n}$

Product of Two Vectors

There are two types of products of vectors:

1. **Scalar (or dot) product:** $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$. Properties include distributivity over addition and commutativity.
2. **Vector (or cross) product:** $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \hat{n}$, where θ is the angle between $\vec{a}$ and $\vec{b}$, and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$, and $\hat{n}$ form a right-handed system. Properties include distributivity over addition, but it is not commutative.

Chapter 11

Three Dimensional Geometry

Introduction to 3D Geometry

Three-dimensional geometry extends the concepts of coordinate geometry to three dimensions. This chapter will explore lines, planes, and their relationships in three-dimensional space using vector methods.

Direction Cosines and Ratios

If a line makes angles α, β, and γ with the positive directions of the x, y, and z axes respectively, then $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the **direction cosines** of the line. They are denoted by l, m, and n respectively. Since $l^2 + m^2 + n^2 = 1$, the direction cosines are unique. Any three numbers proportional to the direction cosines are called **direction ratios** (or **direction numbers**). If a, b, and c are direction ratios, then:

$$l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

The direction cosines of the line segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are:

$$l = \frac{x_2 - x_1}{PQ}, \quad m = \frac{y_2 - y_1}{PQ}, \quad n = \frac{z_2 - z_1}{PQ}$$

where $PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$. The direction ratios are $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$.

Equation of a Line

Vector Equation: Let $\vec{a}$ be the position vector of a point A on the line and $\vec{b}$ be a vector parallel to the line. Then, the vector equation of the line is given by:

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

where $\vec{r}$ is the position vector of any point P on the line, and λ is a scalar parameter. If $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$, then a, b, and c are the direction ratios of the line.
Cartesian Equation: The Cartesian equation of the line passing through a point (x_1, y_1, z_1) and with direction ratios a, b, c is:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

If the line passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2), then its equation is:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Angle Between Lines

Let θ be the angle between two lines with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2. Then:

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$$

If the lines are given by $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$, then the angle between the lines is:

$$\cos\theta = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Two lines are perpendicular if $\cos\theta = 0$, which implies $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$. They are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Shortest Distance Between Skew Lines

If two lines in space are skew (neither parallel nor intersecting), then the shortest distance between them is the length of the line segment perpendicular to both lines.
Let the equations of the two lines be:

$$\vec{r} = \vec{a_1} + \lambda \vec{b_1} \quad \text{and} \quad \vec{r} = \vec{a_2} + \mu \vec{b_2}$$

where $\vec{a_1}$ and $\vec{a_2}$ are the position vectors of points on the lines, and $\vec{b_1}$ and $\vec{b_2}$ are vectors parallel to the lines. The shortest distance d between the lines is given by:

$$d = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

If the lines are given in Cartesian form as:

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \quad \text{and} \quad \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

then the shortest distance is:

$$d = \frac{|(x_2 - x_1)(b_1c_2 - b_2c_1) + (y_2 - y_1)(c_1a_2 - c_2a_1) + (z_2 - z_1)(a_1b_2 - a_2b_1)|}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

If the lines are parallel, the shortest distance is the perpendicular distance between them. Let $\vec{a_1}$ be the position vector of a point on one line and $\vec{a_2}$ be the position vector of a point on the other line, and let $\vec{b}$ be a vector parallel to both lines. Then the shortest distance is:

$$d = \frac{|(\vec{a_2} - \vec{a_1}) \times \vec{b}|}{|\vec{b}|}$$

In cartesian form, if $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ and $\frac{x-x_2}{a} = \frac{y-y_2}{b} = \frac{z-z_2}{c}$ are the equations of the two parallel lines, then the shortest distance between them is given by:

$$d = \frac{|(x_2 - x_1)(b) + (y_2 - y_1)(-a) + (z_2 - z_1)(0)|}{\sqrt{a^2 + b^2}}$$

Chapter 12

Linear Programming

Introduction to Linear Programming

Linear programming deals with optimizing (maximizing or minimizing) a linear objective function subject to linear constraints (inequalities or equations). These problems arise in various fields, including resource allocation, production planning, and transportation. This chapter focuses on solving linear programming problems graphically.

Problem Formulation

A linear programming problem involves:

1. A **linear objective function:** A function to be maximized or minimized (e.g., profit, cost). Typically of the form $Z = ax + by$ for two variables.
2. **Linear constraints:** Inequalities or equations that restrict the values of the variables (e.g., resource limitations, production capacities). These are typically expressed as linear inequalities (e.g., $ax + by \leq c$, $ax + by \geq c$) or equations ($ax + by = c$).
3. **Non-negativity constraints:** The variables are non-negative (e.g., $x \geq 0$, $y \geq 0$). We cannot have negative quantities of goods or resources.

The goal is to find the values of the variables that optimize the objective function while satisfying all the constraints.

Graphical Method

The graphical method is used to solve linear programming problems with two variables. The steps are:

1. **Graph the constraints:** Plot the linear inequalities representing the constraints on a coordinate plane. The feasible region is the area where all constraints are satisfied.
2. **Identify corner points:** Find the coordinates of the vertices (corner points) of the feasible region.
3. **Evaluate the objective function:** Substitute the coordinates of each corner point into the objective function.
4. **Optimal solution:** The corner point that yields the maximum (or minimum) value of the objective function is the optimal solution. If the feasible region is unbounded, the maximum or minimum value might not exist.

If the feasible region is bounded, the optimal solution will always occur at one of the corner points. If it's unbounded, additional analysis may be required.

Chapter 13

Probability

Introduction

In earlier classes, we studied probability as a measure of the uncertainty of events in random experiments. The axiomatic approach, developed by Kolmogorov, treated probability as a function of experimental outcomes. We will build upon the addition rule of probability and explore new concepts.

Conditional Probability

If we have two events, E and F, from the same sample space, the occurrence of one event may affect the probability of the other. The **conditional probability** of event E given that F has occurred is:

$$P(E|F) = \frac{P(E \cap F)}{P(F)}, \quad \text{provided } P(F) \neq 0$$

This formula can be interpreted as the probability of E within the reduced sample space F.

Properties of Conditional Probability

- **Property 1:** $P(S|F) = 1$
- **Property 2:** If A and B are events, and F is an event with $P(F) \neq 0$, then $P((A \cup B)|F) = P(A|F) + P(B|F) - P((A \cap B)|F)$ If A and B are disjoint, $P((A \cup B)|F) = P(A|F) + P(B|F)$.
- **Property 3:** $P(E'|F) = 1 - P(E|F)$

Multiplication Theorem on Probability

For two events E and F:

$$P(E \cap F) = P(F)P(E|F) = P(E)P(F|E)$$

This extends to three or more events:

$$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2)$$

Independent Events

Two events E and F are **independent** if the occurrence of one does not affect the probability of the other. This means:

$$P(E|F) = P(E) \quad \text{and} \quad P(F|E) = P(F)$$

Equivalently, $P(E \cap F) = P(E)P(F)$. This definition extends to more than two events.

Theorem of Total Probability

Let $\{E_1, E_2, \ldots, E_n\}$ be a partition of the sample space S such that $P(E_i) > 0$ for all i. Let A be any event in S. Then:

$$P(A) = \sum_{i=1}^{n} P(E_i)P(A|E_i)$$

Bayes' Theorem

Bayes' theorem calculates the probability of an event E given that event A has occurred:

$$P(E_i|A) = \frac{P(E_i)P(A|E_i)}{\sum_{j=1}^{n} P(E_j)P(A|E_j)}$$

where $E_1, E_2, ..., E_n$ form a partition of the sample space. This theorem is crucial for reversing conditional probabilities.

Random Variables

A **random variable (RV)** is a function that assigns a real number to each outcome in a sample space. For a random variable, if $P(X = x) = p(x)$, then $\sum_x p(x) = 1$.

Expectation (Mean) of a Random Variable

The **expectation** (or expected value or mean) of a random variable provides a measure of its central tendency. It is given by $E[X] = \sum_x xP(X = x)$